

Shooting with Degree Theory: Analysis of Some Weighted Polyharmonic Systems

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Abstract

The author shows the existence of positive solutions to a general class of non-autonomous, semilinear elliptic systems in the whole space including weighted Hardy–Littlewood–Sobolev type systems. The novel method used here implements the classical shooting method enhanced by topological degree theory. First, a target map is constructed which aims the shooting method, then the non-degeneracy conditions are established which are used to guarantee the continuity of the map needed in applying topological degree theory. The existence of zeros for the target map guaranteed by degree theory combined with a Liouville type theorem for the corresponding Dirichlet problem will show the existence of positive solutions to the class of systems.

Keywords: Degree theory, weighted Hardy–Littlewood–Sobolev inequality, polyharmonic equations, shooting method, stationary Schrödinger.

Mathematics Subject Classification: 35B09, 35B33, 35J30, 35J48, 47H11. 47H11—

1 Introduction

A well-known difficulty in studying nonlinear elliptic systems, or any nonlinear system of partial differential equations (PDEs) for that matter, is on developing tools useful in their analysis. Such tools are usually limited to specific problems; that is, a technique for examining one class of problems may prove ineffective in examining other classes of problems. Our main objective of this article is to further develop and refine a novel approach—first introduced in [22]—for proving the existence of

positive solutions and related properties to higher-order, nonlinear system of elliptic equations in the whole space. As we shall see below, the class of problems we examine will include the well-known Lane–Emden and Hardy–Littlewood–Sobolev type systems along with their weighted counterparts as motivating examples. Remarkably, the mathematical tools utilized within this framework are more or less elementary by themselves. The first main result proves the existence of positive solutions, under reasonable assumptions, to the general system

$$(-\Delta)^{k_i} u_i = f_i(|x|, u) \quad \text{in } x \in \mathbb{R}^n \setminus \{0\}, \quad \text{for } i = 1, 2, \dots, L. \quad (1.1)$$

As will be demonstrated below, proving the existence of positive solutions to this system of elliptic PDEs involves reformulating the problem in radial coordinates then applying the classical shooting method coupled with a Liouville type theorem for the corresponding Dirichlet problem. Specifically, a natural ingredient of the proof entails constructing a continuous **target map** which aims the shooting method. Then a fixed point argument via degree theory is invoked to guarantee the existence of zeros of this target map which enables us to identify the correct initial shooting positions for the shooting method. Combining this with a Liouville type theorem will imply the existence of positive, radially symmetric solutions to (1.1).

In establishing our general results, a central and motivating example is the generalized weighted system,

$$\begin{cases} (-\Delta)^{\frac{\gamma}{2}} u = \frac{v^q}{|x|^{\beta_1}} & \text{in } \mathbb{R}^n \setminus \{0\}, \\ (-\Delta)^{\frac{\gamma}{2}} v = \frac{u^p}{|x|^{\beta_2}} & \text{in } \mathbb{R}^n \setminus \{0\}, \\ u, v > 0 & \text{in } \mathbb{R}^n. \end{cases} \quad (1.2)$$

Notice that when $\gamma = 2$ and $\beta_1 = \beta_2 = 0$, we may consider the whole space and this system reduces to the well-known Lane–Emden system,

$$\begin{cases} -\Delta u = v^q, & u > 0 & \text{in } \mathbb{R}^n, \\ -\Delta v = u^p, & v > 0 & \text{in } \mathbb{R}^n, \end{cases} \quad (1.3)$$

or more generally the Hardy–Littlewood–Sobolev (HLS) system when $\gamma > 2$ and $\beta_1 = \beta_2 = 0$,

$$\begin{cases} (-\Delta)^{\frac{\gamma}{2}} u = v^q, & u > 0 & \text{in } \mathbb{R}^n, \\ (-\Delta)^{\frac{\gamma}{2}} v = u^p, & v > 0 & \text{in } \mathbb{R}^n. \end{cases} \quad (1.4)$$

The Lane–Emden and HLS systems have received much attention in the past few decades. For instance, the scalar case was studied in [2, 4, 17], and similar problems have been approached geometrically in [5, 9]. Related systems including its generalized version, the HLS type systems and related problems have been studied as well

[6]–[12], [14]–[15], [19]–[21], and [23, 28, 31, 35]. When $\gamma = 2k$ is an even integer, (1.4) is equivalent to the integral system

$$\begin{cases} u(x) = \int_{\mathbb{R}^n} \frac{v^q(y)}{|x-y|^{n-\gamma}} dy, & u > 0 \text{ in } \mathbb{R}^n, \\ v(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\gamma}} dy, & v > 0 \text{ in } \mathbb{R}^n, \\ 0 < p, q < \infty, \quad 0 < \gamma < n, \end{cases} \quad (1.5)$$

in the sense that a solution of one system, multiplied by a suitable constant if necessary, is also a solution of the other when $p, q > 1$. Hence, systems (1.4) and (1.5) are both referred to as the HLS system. Now when studying the HLS system, the exponents p, q , and γ play an essential role in its behavior. More precisely, there are three important cases to be considered: for the positive constants $0 < p, q < \infty$ and $\gamma = 2k$ ($k \in \mathbb{N}$),

- (a) $\frac{1}{1+p} + \frac{1}{1+q} > \frac{n-2k}{n}$,
- (b) $\frac{1}{1+p} + \frac{1}{1+q} = \frac{n-2k}{n}$, and
- (c) $\frac{1}{1+p} + \frac{1}{1+q} < \frac{n-2k}{n}$.

The HLS system is said to **subcritical** if (a), **critical** if (b), and **supercritical** if (c). In the case of (1.3), the famous Lane–Emden conjecture—an analogue to the celebrated result of Gidas and Spruck [17] in the scalar case—states that this elliptic system with subcritical exponents has no classical solution. This has been completely settled for radial solutions [27, 33], for dimensions $n \leq 4$ [30, 34, 37], and for $n \geq 4$ under certain subregions of subcritical exponents [3, 16, 27, 32, 37, 38]. With the help of the moving planes method in integral forms, the work in [13]—when combined with the non-existence results in [26]—provides a partial resolution of this conjecture as well.

In the critical case, the integral system (1.5) is the Euler–Lagrange equations of the fundamental Hardy–Littlewood–Sobolev inequality. In this case, we call a solution of the above system **ground state** if it ‘optimizes’ the HLS inequality. Recall that the HLS inequality (see [18, 25, 36]) states that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} dx dy \leq C_{s,\lambda,n} \|f\|_r \|g\|_s \quad (1.6)$$

where $0 < \lambda < n$, $1 < s, r < \infty$, $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{n} = 2$, $f \in L^r(\mathbb{R}^n)$, and $g \in L^s(\mathbb{R}^n)$. To find the best constant in the HLS inequality, one maximizes the HLS functional:

$$J(f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\lambda} dx dy \quad (1.7)$$

under the constraints $\|f\|_r = \|g\|_s = 1$. Let $p = \frac{1}{r-1}$, $q = \frac{1}{s-1}$ and with a suitable scaling such as $u = c_1 f^{r-1}$ and $v = c_2 g^{s-1}$, the Euler–Lagrange equations are precisely the system of integral equations in (1.5). Here, $u \in L^{p+1}$ and $v \in L^{q+1}$ are equivalent to $f \in L^r$ and $g \in L^s$ and the exponents, $0 < p, q < \infty$, satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\gamma}{n}.$$

Moreover, Lieb proved in [25] the existence of positive solutions to (1.5) which maximize the corresponding functionals $J(f, g)$ in the class of $u \in L^{p+1}$ and $v \in L^{q+1}$. In other words, there exist maximizers to the Euler–Lagrange equations under critical exponents, therefore showing the existence of ground state solutions for the HLS system. In addition, Hardy and Littlewood also introduced the following double weighted inequality which was later generalized by Stein and Weiss [39]:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} dx dy \leq C_{\alpha, \beta, s, \lambda, n} \|f\|_r \|g\|_s \quad (1.8)$$

where $\alpha + \beta \geq 0$,

$$1 - \frac{1}{r} - \frac{\lambda}{n} < \frac{\alpha}{n} < 1 - \frac{1}{r}, \text{ and } \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{n} = 2.$$

To find the best constant in the double weighted inequality, one maximizes the associated functional:

$$J(f, g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} dx dy.$$

As was shown above, one can show that the corresponding Euler–Lagrange equations are the system of integral equations:

$$\begin{cases} u(x) = \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{v^q(y)}{|y|^\beta |x-y|^\lambda} dy, & u > 0 \text{ in } \mathbb{R}^n, \\ v(x) = \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^\alpha |x-y|^\lambda} dy, & v > 0 \text{ in } \mathbb{R}^n, \end{cases} \quad (1.9)$$

where $0 < p, q < \infty$, $0 < \lambda < n$, $\frac{\alpha}{n} < \frac{1}{p+1} < \frac{\lambda+\alpha}{n}$, and $\frac{1}{1+p} + \frac{1}{1+q} = \frac{\lambda+\alpha+\beta}{n}$. Interestingly, the authors in [11] considered a weighted HLS system. Namely, they proved, under appropriate mild conditions, the uniqueness of solutions to the singular non-linear system,

$$\begin{cases} -\Delta(|x|^\alpha u) = \frac{v^q}{|x|^\beta} & \text{in } \mathbb{R}^n \setminus \{0\}, \\ -\Delta(|x|^\beta v) = \frac{u^p}{|x|^\alpha} & \text{in } \mathbb{R}^n \setminus \{0\}, \end{cases} \quad (1.10)$$

and classified all the solutions for the case $\alpha = \beta$ and $p = q$, thus obtaining the best constant in the corresponding weighted HLS inequality.

Let us remark on the case of supercritical exponents for the HLS system in relation to the work in this article. As a simple illustration, let $u = v$ and $p = q$ in (1.5) to obtain the following scalar integral equation,

$$u(x) = \int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\gamma}} dy, \quad u > 0 \quad \text{in } \mathbb{R}^n, \quad (1.11)$$

with the corresponding partial differential equation

$$(-\Delta)^k u(x) = u^p(x), \quad 2k < n, \quad u > 0 \quad \text{in } \mathbb{R}^n. \quad (1.12)$$

As before, both the integral and differential equations are called supercritical if $p > \frac{n+\gamma}{n-\gamma}$, critical if $p = \frac{n+\gamma}{n-\gamma}$ and subcritical if $p < \frac{n+\gamma}{n-\gamma}$. In the supercritical case with $k = 1$, the shooting method can be successfully applied to (1.12), however, much difficulty arises even in the scalar case for $k \geq 2$. In the results of this paper, we circumvent these difficulties by enhancing the shooting method via degree theory in an interesting way and the approach used applies to systems as well. So the primary objective of this paper is to further develop our framework to handle even more general systems such as the weighted system (1.2) especially since such existence results are not so well developed for these problems. Hence, we shall determine the conditions on such systems which allow us to prove existence of solutions using our technique, and in doing so, we demonstrate how to handle even the case of (1.2) which is not included in the results of [22]. In addition to the Liouville type results, the difficulty in implementing our technique lies in determining the sufficient conditions which guarantee the continuity of the target map. This difficulty motivates our consideration of **non-degeneracy** conditions. Specifically, we introduce non-degeneracy conditions each geared to handle systems such as (1.2) with varying exponents and weights.

The rest of this manuscript is structured as follows. In section 2, we introduce some preliminary definitions and the precise statements of our main results. Section 3 gives the proofs of the first existence theorem concerning the general system (1.1). In section 4, we prove the existence theorems dealing with the weighted system (1.2). In order to prove this theorem, a Liouville type theorem for this system is required; thus such a non-existence result for the corresponding Dirichlet problem is also provided in this section.

2 Preliminaries and Main Results

Consider the system

$$\begin{cases} (-\Delta)^{k_i} u_i = f_i(|x|, u), & \text{in } \mathbb{R}^n \setminus \{0\}, \\ u_i > 0, & \text{in } \mathbb{R}^n, \quad \text{for } i = 1, 2, \dots, L. \end{cases} \quad (2.1)$$

with the following assumptions. From this point on, we will always assume $k_i \geq 1$ and $F(|x|, u) = (f_1(|x|, u), f_2(|x|, u), \dots, f_L(|x|, u))$ satisfies the following conditions:

- (a) $F : \mathbb{R}_+ \times \mathbb{R}_+^L \mapsto \mathbb{R}_+^L$ is a continuous vector-valued map,
- (b) $F(|x|, u)$ is locally Lipschitz continuous in the second argument uniformly in the interior of $\mathbb{R}_+ \times \mathbb{R}_+^L$.

Non-degeneracy condition 1. Given a $v \in \partial\mathbb{R}_+^n$, let I_v^0 and I_v^+ contain the components $j \in \{1, 2, \dots, L\}$ of v such that $v_j = 0$ and $v_j > 0$, respectively. Then there are constants $\lambda = \lambda(v) > 0$ and $\beta > -2$ and a $\delta = \delta(v)$ such that if $|v - w| < \delta$ then

$$\sum_{j \in I_v^0} f_j(|x|, w) \geq \lambda(v)|x|^\beta \text{ for } |x| \ll 1.$$

Remark 2.1. In the case where $k_i = 1$ for all $1 \leq i \leq L$, we shall always assume that $f_{i_0}(|x|, u) = f_{i_0}(u)$ for some $1 \leq i_0 \leq L$.

Non-degeneracy condition 2. Suppose that $k_i = 1$ for all i , $F(|x|, v) = F(v)$, and $F(v) \neq 0$ whenever $v > 0$ in (2.1). Given any two real numbers $0 \leq m < M$, we have for $v = (v_1, v_2, \dots, v_L) \in \mathbb{R}_+^L$ such that

$$v_{i_k} \leq m \text{ for } k = 1, 2, \dots, j \text{ and } m < v_{i_k} \leq M \text{ for } k = j + 1, \dots, L,$$

where $\{i_1, i_2, \dots, i_L\}$ is any permutation of $\{1, 2, \dots, L\}$, there exists a constant $C_{m,M} > 0$ such that

$$\max_{j+1 \leq k \leq L} f_{i_k}(v) \leq \frac{C_{m,M}}{L} \sum_{k=1}^j f_{i_k}(v). \quad (2.2)$$

Moreover, if the right-hand side of (2.2) is zero, then the left-hand side must also vanish simultaneously.

Systems of the form (1.1) satisfying conditions (a)–(b) and the non-degeneracy ‘condition 1’ (or ‘condition 2’) are said to be **non-degenerate ‘type I’ (or ‘type II’)**. The weighted HLS system is an example of a non-degenerate type I system and the system,

$$\begin{cases} -\Delta u = u^s v^q, & u > 0 & \text{in } \mathbb{R}^n, \\ -\Delta v = v^t u^p, & v > 0 & \text{in } \mathbb{R}^n, \\ u, v > 0 & & \text{in } \mathbb{R}^n, \end{cases} \quad (2.3)$$

is an example of a non-degenerate type II system provided $p \geq t \geq 0$, and $q \geq s \geq 0$, but also notice that the latter is not of type I.

The first theorem presented in this paper illustrates how the existence of solutions for non-degenerate systems (1.1) follows from the non-existence of solutions to the corresponding Dirichlet problem,

$$\begin{cases} (-\Delta)^{k_i} u_i = f_i(|x|, u) & \text{in } B_R(0) \setminus \{0\}, \\ u_i > 0 & \text{in } B_R(0), \\ u_i = -\Delta u_i = \dots = (-\Delta)^{k_i-1} u_i = 0 & \text{on } \partial B_R(0), \ i = 1, 2, \dots, L, \end{cases} \quad (2.4)$$

for all $R > 0$. Here $B_R(0)$ denotes the open ball of radius R centered at the origin and sometimes we use B_R instead for brevity. We will be mainly concerned with type I systems in this paper, but the authors in [24] obtained analogous results for unweighted, non-degenerate systems of type II. Namely, as a motivating example, one of the results in that work proved the following:

Theorem. *The non-degenerate system (2.3) has a solution of class $C^2(\mathbb{R}^n)$ provided that $q \geq t \geq 0$, $p \geq s \geq 0$, and*

$$\frac{1}{1+q} + \frac{1}{1+p} \leq \frac{n-2}{2}.$$

Now we are ready to state our main results.

Theorem 1. *The non-degenerate type I system (2.1) admits a radially symmetric classical solution provided that (2.4) admits no radially symmetric classical solution for all $R > 0$.*

This theorem has the following consequence.

Theorem 2. *The system*

$$\begin{cases} (-\Delta)^k u = \frac{v^q}{|x|^{\beta_1}} & \text{in } \mathbb{R}^n \setminus \{0\}, \\ (-\Delta)^k v = \frac{u^p}{|x|^{\beta_2}} & \text{in } \mathbb{R}^n \setminus \{0\}, \\ u, v > 0 & \text{in } \mathbb{R}^n, \end{cases} \quad (2.5)$$

admits a solution of class $C^{2k}(\mathbb{R}^n \setminus \{0\})$ provided that

$$\frac{n-\beta_1}{1+q} + \frac{n-\beta_2}{1+p} \leq n-2k,$$

where $0 < p, q < \infty$ and $\beta_1, \beta_2 \in (-\infty, 2)$.

In addition, we have the following Liouville type result which will be important in proving Theorem 2. Notice that we can show non-existence of solutions to an even more general system than the weighted HLS system.

Theorem 3. *The system,*

$$\begin{cases} (-\Delta)^k u = \frac{u^s v^q}{|x|^{\beta_1}} & \text{in } B_R(0) \setminus \{0\}, \\ (-\Delta)^k v = \frac{v^t u^p}{|x|^{\beta_2}} & \text{in } B_R(0) \setminus \{0\}, \\ u, v > 0 & \text{in } B_R(0), \\ u = -\Delta u = \dots = (-\Delta)^{k-1} u = 0 & \text{on } \partial B_R(0), \\ v = -\Delta v = \dots = (-\Delta)^{k-1} v = 0 & \text{on } \partial B_R(0), \end{cases} \quad (2.6)$$

admits no radially symmetric solution of class $C^{2k}(B_R(0) \setminus \{0\}) \cap C^{2k-1}(\overline{B_R(0)})$ for any $R > 0$ provided that

$$\frac{n - \beta_1}{1 + q} + \frac{n - \beta_2}{1 + p} \leq n - 2k \quad (2.7)$$

where $s, t, p, q \geq 0$ and $\beta_1, \beta_2 \in (-\infty, n)$.

3 Proof of Theorem 1

In order to prove the first two theorems we must introduce several key ideas and lemmas. As mentioned earlier, the proof centers on constructing a map which aims the shooting method. A crucial step will be to show the continuity of this map, and the non-degeneracy conditions are exactly what is needed in guaranteeing continuity.

For $i = 1, 2, \dots, L$ set $w_{i,j} = (-\Delta)^{j-1}u_i$, $1 \leq j \leq k_i$ and consider the system

$$\begin{cases} -\Delta w_{i,1} = w_{i,2}, \\ -\Delta w_{i,2} = w_{i,3}, \\ \vdots \\ -\Delta w_{i,k_i-1} = w_{i,k_i}, \\ -\Delta w_{i,k_i} = f_i(|x|, w_{1,1}, w_{2,1}, \dots, w_{L,1}) & \text{in } \mathbb{R}^n \setminus \{0\}, \\ w_{i,1}, w_{i,2}, \dots, w_{i,k_i} > 0 & \text{in } \mathbb{R}^n, \\ \text{where } i = 1, 2, \dots, L. \end{cases} \quad (3.1)$$

Solutions to (3.1) are clearly solutions to (2.1), so it will suffice to show the existence of solutions to (3.1) instead. We can express the above system into the form:

$$\begin{cases} -\Delta w_1 = f_1(r, w), -\Delta w_2 = f_2(r, w), \\ -\Delta w_3 = f_4(r, w), \dots, -\Delta w_{L-1} = f_{L-1}(r, w), \\ -\Delta w_L = f_L(r, w) & \text{in } \mathbb{R}^n \setminus \{0\}, \\ w_1, w_2, \dots, w_L > 0 & \text{in } \mathbb{R}^n, \end{cases} \quad (3.2)$$

where we still use L to represent a generic positive integer. We shall work with (3.2) instead when proving Theorem 1, but note that the non-degeneracy condition 1 still holds true for this new system. Now let us define the aforementioned target map. For any strictly positive initial value $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_L)$, consider the IVP

$$\begin{cases} w_i''(r) + \frac{n-1}{r}w_i'(r) = -f_i(r, w(r)), \\ w_i'(0) = 0, \quad w_i(0) = \alpha_i \text{ for } i = 1, 2, \dots, L. \end{cases} \quad (3.3)$$

Clearly (3.3) is equivalent to (3.2) in radial coordinates.

Definition 3.1. Define the target map $\psi : \mathbb{R}_+^L \mapsto \mathbb{R}_+^L$ as follows. For $\alpha \in \text{int}(\mathbb{R}_+^L)$, the interior of \mathbb{R}_+^L ,

- (a) $\psi(\alpha) = w(r_0)$ where r_0 is the smallest such r for which $w_{i_0}(r) = 0$ for some $1 \leq i_0 \leq L$,
- (b) otherwise, if no such r_0 exists, then $\psi(\alpha) = \lim_{r \rightarrow \infty} w(r)$.
- (c) Moreover, $\psi \equiv \text{Identity}$ on $\partial \mathbb{R}_+^L$.

Remark 3.2. We may think of (a) as the case when the solution hits the wall for the first time and (b) is the case where it never hits the wall. Observe also that ψ is equivalent to the identity map on the wall. This property is crucial when we apply the tools from topological degree theory.

The next lemma is a standard result from Brouwer topological degree theory and can be found in various literature, see [1] and [29] for instance.

Lemma 3.3 (Dependence on boundary values). *Let $U \subset \mathbb{R}^n$ be a bounded open set and $f, g : \overline{U} \mapsto \mathbb{R}^n$ are continuous maps. Suppose that $f \equiv g$ on ∂U and $a \notin f(\partial U) = g(\partial U)$, then $\text{degree}(f, U, a) = \text{degree}(g, U, a)$.*

One may recall the important property that if $\text{degree}(f, U, a) \neq 0$, then there exists a point $x \in U$ such that $f(x) = a$.

Lemma 3.4. *The target map $\psi : \mathbb{R}_+^L \mapsto \partial \mathbb{R}_+^L$ is continuous.*

Proof of Lemma 1. Choose any $\overline{\alpha} \in \mathbb{R}_+^L$ and fix an $\epsilon > 0$. Without loss of generality, there are two cases to consider:

- (1) $\overline{\alpha} \in \partial \mathbb{R}_+^L$,
- (2) $\overline{\alpha} \in \text{int}(\mathbb{R}_+^L)$ and $\psi(\overline{\alpha}) = w(r_0, \overline{\alpha})$,

We only consider the two cases since the remaining case where $\overline{\alpha} \in \text{int}(\mathbb{R}_+^L)$ with $w(r, \alpha) > 0$ for all $r > 0$ says that we have found an initial shooting position $\overline{\alpha}$ that gives us a desired global solution as stated in the main theorem.

Case (1): By definition we have that $\psi(\overline{\alpha}) = \overline{\alpha}$ and fix $\epsilon > 0$. We can find a $\delta_1 > 0$ such that for $|\alpha - \overline{\alpha}| < \delta_1 = \delta_1(\overline{\alpha})$, the non-degeneracy condition 1 implies that

$$\sum_{j \in I_{\alpha}^0} f_j(r, \alpha) \geq \lambda(\overline{\alpha}) r^{\beta} \quad \text{for all } r \ll 1.$$

Then we can find $\delta_2 > 0$ such that $|\bar{\alpha} - w(r, \alpha)| < \delta_1$ for $r < \delta_2$ and $|\alpha - \bar{\alpha}| < \delta_2$.
Let

$$W^0(r, \alpha) := \sum_{j \in I_{\bar{\alpha}}^0} w_j(r, \alpha).$$

Then the non-degeneracy condition 1 and (3.3) imply that

$$-\frac{d}{dr} \left(r^{n-1} \frac{dW^0}{dr}(r, \alpha) \right) \geq \lambda(\bar{\alpha}) r^{n-1+\beta}.$$

Integrating this twice with respect to r yields

$$W^0(r, \alpha) \leq \left(\sum_{j \in I_{\bar{\alpha}}^0} \alpha_j \right) - \frac{\lambda(\bar{\alpha})}{(2+\beta)(n+\beta)} r^{2+\beta}.$$

From this we can find $\delta > 0$ sufficiently small with $\delta < \epsilon$ such that $W^0(r_\alpha, \alpha) = 0$ for some $r_\alpha < \delta_2$ and

$$|\psi(\alpha) - \psi(\bar{\alpha})| = |\psi(\alpha) - \bar{\alpha}| \leq |w(r_\alpha, \alpha) - \bar{\alpha}| \leq \delta < \epsilon$$

whenever $|\alpha - \bar{\alpha}| < \delta$.

Case (2): Since the source terms f_i are non-negative, $u'_{i_0}(r_0, \bar{\alpha}) < 0$ by a direct computation or simply by Hopf's Lemma. This transversality condition along with the ODE stability imply that for α sufficiently close to $\bar{\alpha}$, the solution to this perturbed IVP must hit the wall and $\psi(\alpha)$ must be close to $\psi(\bar{\alpha})$.

This completes the proof that ψ is continuous at $\bar{\alpha} \in \mathbb{R}_+^L$. □

Lemma 3.5. *For every $a > 0$, there exists an $\alpha_a \in A_a$ where*

$$A_a := \left\{ \alpha \in \mathbb{R}_+^L \mid \sum_{i=1}^L \alpha_i = a \right\}$$

such that $\psi(\alpha_a) = 0$.

Proof of Lemma 2. Define the set B_a as follows

$$B_a := \left\{ \alpha \in \partial \mathbb{R}_+^L \mid \sum_{i=1}^L \alpha_i \leq a \right\}.$$

It follows that ψ maps A_a into B_a due to the non-increasing property of solutions. Now define the continuous map $\phi : B_a \rightarrow A_a$ by

$$\phi(\alpha) = \alpha + \frac{1}{L} \left(a - \sum_{i=1}^L \alpha_i \right) (1, 1, \dots, 1)$$

with continuous inverse $\phi^{-1} : A_a \longrightarrow B_a$ defined

$$\phi^{-1}(\alpha) = \alpha - \left(\min_{i=1, \dots, L} \alpha_i \right) (1, 1, \dots, 1).$$

Set $\eta = \phi \circ \psi : A_a \longrightarrow A_a$. Then η is continuous on A_a and is equivalent to the identity map on the boundary of A_a . By Lemma 3.3, the index of the map satisfies $\text{degree}(\eta, A_a, \alpha) = \text{degree}(\text{Identity}, A_a, \alpha) = 1 \neq 0$ for any interior point of A_a . So η is onto, and thus ψ is onto. Then there exists an $\alpha_a \in A_a$ such that $\psi(\alpha_a) = 0$. \square

Proof of Theorem 1: Let $w = w(r)$ be a solution of (3.3) with initial position $w(0) = \alpha_a$ as guaranteed by Lemma 3.5 for fixed a . This solution must never hit the wall. If this was the case, then there would be a smallest finite value $r = r_0$ such that $w(r_0) = \psi(\alpha_a) = 0$. But this would imply that $w = w(|x|)$ is a radially symmetric solution to (2.4) with $R = r_0$, which is a contradiction. Hence, $w = w(|x|)$ must be a radially symmetric solution to (2.1).

Remark 3.6. *In the proof of Theorem 1, we are implicitly using the fact that the Dirichlet problem (2.4) is equivalent, under classical solutions, to the Dirichlet problem,*

$$\left\{ \begin{array}{ll} -\Delta w_{i,1} = w_{i,2}, \\ -\Delta w_{i,2} = w_{i,3}, \\ \vdots \\ -\Delta w_{i,k_i-1} = w_{i,k_i}, \\ -\Delta w_{i,k_i} = f_i(|x|, w_{1,1}, w_{2,1}, \dots, w_{L,1}) & \text{in } B_R(0) \setminus \{0\}, \\ w_{i,1}, w_{i,2}, \dots, w_{i,k_i} > 0 & \text{in } B_R(0), \\ w_{i,1} = w_{i,2} = \dots = w_{i,k_i} = 0 & \text{on } \partial B_R(0), \end{array} \right.$$

where $w_{i,j} = (-\Delta)^{j-1} u_i$ for $j = 1, 2, \dots, k_i$ and $i = 1, 2, \dots, L$. This equivalence follows from an inductive argument by exploiting the non-negative property of the source term F , the boundary conditions, and the strong maximum principle. We shall revisit this property in the proof of Theorem 3.

4 The Weighted System

The system (1.2) can be reduced into the system

$$\left\{ \begin{array}{ll} -\Delta w_1 = w_2, \dots, -\Delta w_{k-1} = w_k, \\ -\Delta w_k = \frac{w_1^s w_{k+1}^q}{|x|^{\beta_1}}, \\ -\Delta w_{k+1} = w_{k+2}, \dots, -\Delta w_{2k-1} = w_{2k}, \\ -\Delta w_{2k} = \frac{w_{k+1}^t w_1^p}{|x|^{\beta_2}} & \text{in } \mathbb{R}^n \setminus \{0\}, \\ w_1, w_2, \dots, w_{2k} > 0 & \text{in } \mathbb{R}^n, \end{array} \right. \quad (4.1)$$

where $w_1 := u$ and $w_{k+1} := v$. Let $B_R := B_R(0)$. From Theorem 1 it suffices to show the non-existence of solutions to the system (4.2) given below but under Dirichlet boundary conditions on balls. That is, we will show that the system

$$\left\{ \begin{array}{ll} -\Delta w_1 = w_2, \\ \vdots \\ -\Delta w_{k-1} = w_k, -\Delta w_k = \frac{w_1^s w_{k+1}^q}{|x|^{\beta_1}}, \\ -\Delta w_{k+1} = w_{k+2}, \\ \vdots \\ -\Delta w_{2k-1} = w_{2k}, -\Delta w_{2k} = \frac{w_{k+1}^t w_1^p}{|x|^{\beta_2}} & \text{in } B_R \setminus \{0\} \\ w_1, w_2, \dots, w_{2k} > 0 & \text{in } B_R \\ w_1 = w_2 = \dots = w_{2k} = 0 & \text{on } \partial B_R \end{array} \right. \quad (4.2)$$

admits no solution of class $C^2(B_R \setminus \{0\}) \cap C^1(\overline{B_R})$ for any $R > 0$. Observe that the usual Liouville type results such as those found in [27] and [31] do not apply since the source terms f_i in (4.2) do not follow from a potential function so that a variational approach cannot be used. Therefore a result for the non-existence concerning this weighted system, Theorem 3, is provided in order to bypass this issue. First, we need the following lemma.

Lemma 4.1. *Let B_R again denote the ball in \mathbb{R}^n of radius R centered at the origin, and let w_j ($j = 1, 2, \dots, 2k$) solve (4.2). Then*

$$\begin{aligned} \int_{B_R} \frac{u^s v^{q+1}}{|x|^{\beta_1}} dx &= \int_{B_R} \frac{v^t u^{p+1}}{|x|^{\beta_2}} dx \\ &= \int_{B_R} \nabla w_j \cdot \nabla w_{2k+1-j} dx \end{aligned}$$

$$= \int_{B_R} w_{j+1} w_{2k+1-j} dx =: E. \quad (4.3)$$

Proof. To prove this lemma, multiply the $2k$ -th equation in (4.2) by w_1 then integrate over B_R . The repeated application of integration by parts along with the boundary conditions yield

$$\begin{aligned} \int_{B_R} \frac{w_{k+1}^t w_1^{p+1}}{|x|^{\beta_2}} dx &= \int_{B_R} \nabla w_1 \cdot \nabla w_{2k} dx \\ &= - \int_{B_R} w_{2k} \Delta w_1 dx = \int_{B_R} w_{2k} w_2 \\ &= - \int_{B_R} w_2 \Delta w_{2k-1} dx = \int_{B_R} \nabla w_2 \cdot \nabla w_{2k-1} dx \\ &= - \int_{B_R} w_{2k-1} \Delta w_2 dx = \int_{B_R} w_{2k-1} w_3 dx \\ &\vdots \\ &= \int_{B_R} \nabla w_k \cdot \nabla w_{k+1} dx = - \int_{B_R} w_{k+1} \Delta w_k dx \\ &= \int_{B_R} \frac{w_1^s w_{k+1}^{q+1}}{|x|^{\beta_1}} dx. \end{aligned}$$

□

Remark 4.2. Let us be more precise in the calculations found in our proof of Lemma 4.1 since we will be using similar calculations below. For instance, when we multiply, say, the $2k$ -th equation $-\Delta w_{2k} = w_{k+1}^t w_1^p$ with w_1 then integrate over the ball B_R , this should be understood implicitly in the following way. We integrate over $B_R \setminus B_\epsilon(0)$ for $0 < \epsilon < R$ and use an integration by parts to obtain

$$\begin{aligned} \int_{B_R \setminus B_\epsilon(0)} \frac{w_{k+1}^t w_1^{p+1}}{|x|^{\beta_2}} dx &= - \int_{B_R \setminus B_\epsilon(0)} w_1 \Delta w_{2k} dx \\ &= - \int_{\partial B_\epsilon(0)} w_1 \frac{\partial w_{2k}}{\partial \nu} ds + \int_{B_R \setminus B_\epsilon(0)} \nabla w_1 \cdot \nabla w_{2k} dx, \end{aligned}$$

where ν is the inward unit normal vector along $\partial B_\epsilon(0)$. Taking the limit as ϵ tends to zero, the surface integral will vanish since the w_i 's are of the class $C^1(\overline{B_R(0)})$ and are therefore uniformly bounded on $\overline{B_R}$ along with their first-order derivatives. Then we obtain

$$\int_{B_R} \frac{w_{k+1}^t w_1^{p+1}}{|x|^{\beta_2}} dx = \int_{B_R} \nabla w_1 \cdot \nabla w_{2k} dx.$$

All such calculations including those found in the proof of Theorem 3 below should be understood in this way.

Proof of Theorem 3. The equivalence of (2.6) and (4.2) under classical solutions follows from the maximum principle. To see this, first observe that if $u = w_1$ and $v = w_{k+1}$ where w_i 's satisfy (4.2), then u and v must also satisfy (2.6). Now suppose u and v satisfy (2.6) and let $w_i = (-\Delta)^{i-1}u$ for $i = 1, 2, \dots, k$ and $w_i = (-\Delta)^{i-k-1}v$ for $i = k+1, k+2, \dots, 2k$. We only need to show the super polyharmonic property: w_i 's > 0 in B_R . Since it is already given that w_1 and w_{k+1} are positive in B_R , we have that $-\Delta w_k, -\Delta w_{2k} > 0$ in B_R from (4.2). The Dirichlet boundary conditions along with the strong maximum principle imply that $w_k, w_{2k} > 0$ in B_R . Then it follows from (4.2) that $-\Delta w_{k-1}, -\Delta w_{2k-1} > 0$ in B_R . We can repeat the same argument to show that $w_{k-1}, w_{2k-1} > 0$ in B_R . In fact, we may inductively repeat this argument to show the remaining w_i 's are positive in B_R , thus completing our verification that the two systems are equivalent. So with this in mind, it will suffice to show that system (4.2) admits no solution of class $C^2(B_R \setminus \{0\}) \cap C^1(\overline{B_R})$ under the constraint (2.7) with $s, t, p, q \geq 0$.

For $j = 2, 3, \dots, k-1$, multiply the j -th equation in (4.2) by $x \cdot \nabla w_{2k+1-j}$, integrate over B_R , then integrate by parts to obtain

$$\begin{aligned} & - \int_{\partial B_R} \frac{\partial w_j}{\partial n} \frac{\partial w_{2k+1-j}}{\partial n} (x \cdot n) ds + \int_{B_R} \nabla w_j \cdot \nabla w_{2k+1-j} dx \\ & + \int_{B_R} x \cdot w_{j,x_i} \nabla (w_{2k+1-j})_{x_i} dx = \int_{B_R} w_{j+1} (x \cdot \nabla w_{2k+1-j}) dx, \end{aligned} \quad (4.4)$$

where n is the outward pointing unit normal vector. Multiply the $(2k+1-j)$ -th equation in (4.2) by $x \cdot \nabla w_j$ and integrate over B_R and perform analogous calculations as was done in obtaining (4.4). Then summing the resulting equation with (4.4) and using the identity,

$$\begin{aligned} & \int_{B_R} x \cdot w_{j,x_i} \nabla (w_{2k+1-j})_{x_i} + x \cdot w_{2k+1-j,x_i} \nabla (w_j)_{x_i} dx \\ & = \int_{B_R} x \cdot \nabla (\nabla w_j \cdot \nabla w_{2k+1-j}) dx \\ & = -n \int_{B_R} \nabla w_j \cdot \nabla w_{2k+1-j} dx + \int_{\partial B_R} \frac{\partial w_j}{\partial n} \frac{\partial w_{2k+1-j}}{\partial n} (x \cdot n) ds, \end{aligned}$$

we obtain

$$\begin{aligned} & - \int_{\partial B_R} \frac{\partial w_j}{\partial n} \frac{\partial w_{2k+1-j}}{\partial n} (x \cdot n) ds + (2-n) \int_{B_R} \nabla w_j \cdot \nabla w_{2k+1-j} dx \\ & = \int_{B_R} w_{j+1} (x \cdot \nabla w_{2k+1-j}) + w_{2k+2-j} (x \cdot \nabla w_j) dx. \end{aligned} \quad (4.5)$$

Now multiply the $2k$ -th equation in (4.2) by $x \cdot \nabla w_1$ and integrate over B_R to obtain

$$\underbrace{- \int_{B_R} (x \cdot \nabla w_1) \Delta w_{2k} dx}_{:=I_1} = \underbrace{\int_{B_R} (x \cdot \nabla w_1) \frac{w_{k+1}^t w_1^p}{|x|^{\beta_2}} dx}_{:=I_2}.$$

Let us calculate I_1 and I_2 . Using integration by parts,

$$I_1 = - \int_{\partial B_R} \frac{\partial w_1}{\partial n} \frac{\partial w_{2k}}{\partial n} (x \cdot n) ds + \int_{B_R} \nabla w_1 \cdot \nabla w_{2k} dx + \int_{B_R} x_i \frac{\partial w_{2k}}{\partial x_j} \left(\frac{\partial^2 w_1}{\partial x_j \partial x_i} \right) dx,$$

and

$$\begin{aligned} I_2 &= \frac{1}{1+p} \int_{B_R} x_i \frac{w_{k+1}^t (w_1^{p+1})_{x_i}}{|x|^{\beta_2}} \\ &= - \frac{n - \beta_2}{1+p} \int_{B_R} \frac{w_{k+1}^t w_1^{p+1}}{|x|^{\beta_2}} dx - \frac{t}{1+p} \int_{B_R} \frac{w_{k+1}^{t-1} w_1^{p+1}}{|x|^{\beta_2}} (x \cdot \nabla w_{k+1}) dx \end{aligned}$$

Now multiply the first equation by $x \cdot \nabla w_{2k}$ and integrate over B_R to obtain

$$\underbrace{- \int_{B_R} (x \cdot \nabla w_{2k}) \Delta w_1 dx}_{:= II_1} = \underbrace{\int_{B_R} (x \cdot \nabla w_{2k}) w_2 dx}_{:= II_2}.$$

We use integration by parts to rewrite II_1 as follows.

$$II_1 = - \int_{\partial B_R} \frac{\partial w_{2k}}{\partial n} \frac{\partial w_1}{\partial n} (x \cdot n) ds + \int_{B_R} \nabla w_1 \cdot \nabla w_{2k} dx + \int_{B_R} x_i \frac{\partial w_1}{\partial x_j} \left(\frac{\partial^2 w_{2k}}{\partial x_j \partial x_i} \right) dx.$$

By summing together the two equations $I_1 = I_2$ and $II_1 = II_2$ and using the fact that

$$\int_{B_R} x \cdot \nabla (\nabla w_1 \cdot \nabla w_{2k}) dx = \int_{\partial B_R} \frac{\partial w_{2k}}{\partial n} \frac{\partial w_1}{\partial n} (x \cdot n) ds - n \int_{B_R} \nabla w_1 \cdot \nabla w_{2k} dx,$$

we obtain the identity

$$\begin{aligned} (2-n) \int_{B_R} \nabla w_{2k} \cdot \nabla w_1 dx &+ \frac{n - \beta_2}{1+p} \int_{B_R} \frac{w_{k+1}^t w_1^{p+1}}{|x|^{\beta_2}} dx \\ &= \int_{\partial B_R} \frac{\partial w_{2k}}{\partial n} \frac{\partial w_1}{\partial n} (x \cdot n) ds + \int_{B_R} w_2 (x \cdot \nabla w_{2k}) dx \\ &\quad - \frac{t}{1+p} \int_{B_R} \frac{w_1^{p+1} w_{k+1}^{t-1}}{|x|^{\beta_2}} (x \cdot \nabla w_{k+1}) dx. \end{aligned} \tag{4.6}$$

Multiply the k -th and $(k+1)$ -th equations in (4.2) by $x \cdot \nabla w_{k+1}$ and $x \cdot \nabla w_k$, respectively, and integrate over B_R . Using similar calculations to those used in deriving (4.6), we obtain

$$\begin{aligned} (2-n) \int_{B_R} \nabla w_k \cdot \nabla w_{k+1} dx &+ \frac{n - \beta_1}{1+q} \int_{B_R} \frac{w_1^s w_{k+1}^{q+1}}{|x|^{\beta_1}} dx \\ &= \int_{\partial B_R} \frac{\partial w_k}{\partial n} \frac{\partial w_{k+1}}{\partial n} (x \cdot n) ds + \int_{B_R} w_{k+2} (x \cdot \nabla w_k) dx \end{aligned} \tag{4.7}$$

$$-\frac{s}{1+q} \int_{B_R} \frac{w_{k+1}^{q+1} w_1^{s-1}}{|x|^{\beta_1}} (x \cdot \nabla w_1) dx.$$

Observe also that integrating by parts and using the boundary conditions, we see that

$$\begin{aligned} \int_{B_R} x \cdot (w_{j+1} \nabla w_{2k+1-j} + w_{2k+1-j} \nabla w_{j+1}) dx &= \int_{B_R} x \cdot \nabla (w_{j+1} w_{2k+1-j}) dx \\ &= -n \int_{B_R} w_{j+1} w_{2k+1-j} dx. \end{aligned}$$

Using this identity and summing (4.5) over $j = 2, 3, \dots, k-1$ along with (4.6) and (4.7), we see that

$$\begin{aligned} (2-n) \sum_{j=1}^k \int_{B_R} \nabla w_j \cdot \nabla w_{2k+1-j} dx &+ \frac{n-\beta_1}{1+q} \int_{B_R} \frac{w_1^s w_{k+1}^{q+1}}{|x|^{\beta_1}} dx + \frac{n-\beta_2}{1+p} \int_{B_R} \frac{w_{k+1}^t w_1^{p+1}}{|x|^{\beta_2}} dx \\ &+ \sum_{j=1}^{k-1} n \int_{B_R} w_{j+1} w_{2k+1-j} dx \\ &= \sum_{j=1}^k \int_{\partial B_R} \frac{\partial w_j}{\partial n} \frac{\partial w_{2k+1-j}}{\partial n} (x \cdot n) ds - \frac{s}{1+q} \int_{B_R} \frac{w_{k+1}^{q+1} w_1^{s-1}}{|x|^{\beta_1}} (x \cdot \nabla w_1) dx \\ &\quad - \frac{t}{1+p} \int_{B_R} \frac{w_1^{p+1} w_{k+1}^{t-1}}{|x|^{\beta_2}} (x \cdot \nabla w_{k+1}) dx. \end{aligned}$$

Observe that the right hand side of this inequality must be strictly positive by the non-increasing property of the positive radial solutions. Hence, Lemma 4.1 implies that

$$\left\{ k(2-n) + \frac{n-\beta_1}{1+q} + \frac{n-\beta_2}{1+p} + (k-1)n \right\} \cdot E > 0.$$

In other words, we have

$$\frac{n-\beta_1}{1+q} + \frac{n-\beta_2}{1+p} > n-2k,$$

but this contradicts with (2.7). \square

Proof of Theorem 2. The continuity of the target map guaranteed by the non-degeneracy condition along with the Liouville type result of Theorem 3 with $s = t = 0$ will imply the desired result as a consequence of Theorem 1. \square

Remark 4.3. *In the autonomous case where $F(|x|, u) = F(u)$ with the additional assumption that $u = 0$ whenever $F(u) = 0$, the definition of the target map implies that the positive solutions will vanish at infinity. Namely, basic elliptic theory implies that $F(\psi(\alpha_a)) = 0$, so that $\psi(\alpha_a) = 0$ from this extra condition. In other words, the global positive solutions given by our results must exhibit the following asymptotic behavior:*

$$u_i \longrightarrow 0 \text{ uniformly as } |x| \longrightarrow \infty, \quad \text{for } i = 1, 2, \dots, L. \quad (4.8)$$

The desire for similar types of asymptotic behavior is precisely why certain assumptions were placed on F in the non-degeneracy conditions; that is, we required such conditions so that in the case where the positive radial solution never hits the wall, at least one component of the solution decays uniformly to zero and the target map takes values on the wall. This asymptotic property was key in showing the target map was onto, as well.

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